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Self-similarity of quasilattices in two dimensions: I. The n-gonal quasilattice

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Abstract. Self-similarity of a two-dimensional quasilattice with an n-gonal point symmetry has been investigated for even n ($n \geqslant 8$). It is shown that for any n, an n-gonal quasilattice has a self-similarity characterised by a complex number τ; on inflation the quasilattice is scaled by $|\tau|$ and subsequently rotated by arg τ. τ is a PV unit of the n-cyclotomic field $Q(\zeta)$, $\zeta = \exp(2\pi i/n)$; τ satisfies (i) τ and τ^{-1} are both algebraic integers in $Q(\zeta)$, (ii) $|\tau| > 1$ and (iii) $|\tau'| < 1$ for any conjugate τ' but $\bar{\tau}$ (the complex conjugate) of τ in $Q(\zeta)$. PV units are calculated for every n-gonal quasilattice whose multiplicity, $m = \phi(n/2)$ with ϕ being the Eulerian function, is less than 5; $m = 2$ for $n = 8$, 10, 12, $m = 3$ for $n = 14$, 18 and $m = 4$ for $n = 16$, 20, 24, 30. It is found that a quasilattice has two or more independent scales of self-similarity if its multiplicity is larger than 2.

1. Introduction

Since the discovery of an icosahedral quasicrystal by Schechtman *et al* (1984), there have been published extensive investigations on quasilattices, which are presumed to be structural models of quasicrystals (for a review, see Henley (1987)). A quasilattice is a quasiperiodic lattice with a non-crystallographic point symmetry. A Penrose lattice, a representative quasilattice in two dimensions (2D), is the set of vertices of the Penrose tiling which tiles the plane in terms of two kinds of rhombic tiles (de Bruijn 1981). It has a decagonal symmetry as its macroscopic point symmetry. It has another remarkable structural property, i.e. self-similarity; if vertices belonging to some types of the vertices of the Penrose tiling are retained but others are discarded, we obtain a Penrose lattice (and a tiling associated with it) whose scale is larger by the golden ratio $\tau_G = (1+\sqrt{5})/2$ than that of the original lattice (de Bruijn 1981). Similar inflation rules are known for an octagonal quasilattice in 2D (i.e. the Ammann quasilattice of Grünbaum and Shephard (1986)), dodecagonal ones in 2D (Stampfli 1986, Niizeki and Mitani 1987) and an icosahedral one in 3D (Katz and Duneau 1986).

The three numbers, $1+\sqrt{2}$, $(1+\sqrt{5})/2$ and $2+\sqrt{3}$, which appear as the ratios of self-similarity of octagonal, decagonal and dodecagonal quasilattices, respectively, are units in the respective algebraic field; a number in an algebraic field is a unit if it and its inverse are both algebraic integers. Moreover, the self-similarity ratios are PV numbers; a PV number is a real algebraic integer satisfying (i) that it is larger than one and (ii) the absolute value of any conjugate of it is smaller than one (Salem 1983). The importance of a PV number in the self-similarity of a quasiperiodic pattern has been pointed out by Pleasants (1984).

A general method of constructing a quasilattice with a non-crystallographic point symmetry group G in d dimensions is the projection method or, more exactly, the cut and projection method (Krammer and Neri 1984, Duneau and Katz 1985, Janssen

1986). In this method, we start from an l-dimensional ($l > d$) lattice L whose point group is isomorphous to G. We take a d-dimensional invariant subspace E_d of the l-dimensional Euclidean space E_l against G. A part of L is cut by an appropriate strip being parallel to E_d and, subsequently, the lattice points in the part are projected onto E_d, yielding a d-dimensional quasilattice with point symmetry G. This quasilattice has a self-similarity if there exists a linear transformation of E_d, satisfying (i) that it is volume-conserving, (ii) it is an automorphism of L (a one-to-one mapping onto itself), (iii) it leaves E_d invariant and (iv) it acts on the orthogonal complement of E_d in E_l as a contractive linear transformation (Katz and Duneau 1986, Gähler 1986). Note, however, that the relationship between a quasilattice constructed with the projection method and a quasiperiodic pattern with Pleasants' method has not been established yet.

In this paper, we will show that every two-dimensional quasilattice obtained with the projection method has a self-similarity characterised by a PV unit in the n-cyclotomic field $\boldsymbol{Q}(\zeta)$, $\zeta = \exp(2\pi i/n)$, provided that the point symmetry of the quasilattice is D_n, the dihedral group with order $2n$ or, more simply, the point group of a regular n-gon. We introduce complex PV numbers. If a quasilattice has a self-similarity characterised by a complex PV unit, the inflation consists not only of a dilatation but also of a rotation.

In § 2, we will introduce a higher-dimensional n-gonal lattice, whose point symmetry is isomorphous to the point symmetry D_n in 2D. In § 3, an n-gonal quasilattice in 2D is constructed with the projection method from the n-gonal lattice. In § 4, we will discuss self-similarity of the n-gonal quasilattice. In § 5, 'the complex self-similarity ratios' of n-gonal quasilattices are calculated for even integers such that $n \leqslant 20$, $n = 24$ or 30. In § 6, we investigate self-similarity of the diffraction pattern of an n-gonal quasilattice. Finally in § 7, we discuss related subjects.

2. An n-gonal lattice

The Euclidean space in 2D, E_2, can be identified with the complex plane $\boldsymbol{C}$ and a vector in E_2 with a complex number. Let $\zeta = \exp(2\pi i/n)$ $(\equiv \zeta_n)$, where $n(\geqslant 3)$ is an integer. Then, $1, \zeta, \ldots, \zeta^{n-1}$ represent the n vertices of a regular n-gon centred on the origin. Multiplying a fixed complex number, $\alpha(\neq 0)$, onto $\boldsymbol{C}(\simeq E_2)$ gives rise to a rotation of $\boldsymbol{C}$ by $\arg \alpha$ and a subsequent scale transformation by $|\alpha|$. In particular, a multiplication by ζ is equivalent to a pure rotation by $2\pi/n$. The order of ζ as a transformation of $\boldsymbol{C}$ is n, i.e. $\zeta^k \neq 1$ for $0 < k < n$ but $\zeta^n = 1$.

ζ is an algebraic number whose order as an algebraic number is given by $\phi(n)$, where ϕ is the Eulerian function in number theory. An algebraic field $\boldsymbol{Q}(\zeta)$, generated by ζ is called an n-cyclotomic field (for the properties of n-cyclotomic fields used throughout this paper, see Washington (1982) or an appropriate textbook on number theory of algebraic integers). We can restrict our argument only to the case where n is even because $\boldsymbol{Q}(\zeta_n) = \boldsymbol{Q}(\zeta_{2n})$ if n is odd. $\phi(n) = 4$ for $n = 8$, 10 and 12, $\phi(n) = 6$ for $n = 14$ and 18, $\phi(n) = 8$ for $n = 16$, 20, 24 and 30 and $\phi(n) \geqslant 10$ for other even n. Note that $l = \phi(n)$ is an even integer. ζ satisfies an algebraic equation, $P_n(x) = 0$, where $P_n(x) = c_0 + c_1 x + \ldots + c_{l-1} x^{l-1} + x^l$ is the n-cyclotomic polynomial, where the c_i are integers ($c_0 = 1$). The polynomial is irreducible over $\boldsymbol{Q}$, the rational field. Other roots of $P_n(x) = 0$ are the conjugates of ζ in $\boldsymbol{Q}(\zeta)$. Let $m = l/2$. Then, $m - 1$ of $2m - 1$ conjugates of ζ is written as ζ^k, where $1 < k < n/2$ and k has no common divisors with n. We shall denote them as $\zeta', \zeta'', \ldots, \zeta^{(m-1)}$. Then, the other m conjugates of ζ are given by $\zeta^{(m+k)} = \bar{\zeta}^{(k)}$ (the complex conjugate), $k = 0, 1, \ldots, m-1$. We shall call m the multiplicity of $\boldsymbol{Q}(\zeta)$. Note that $\theta_k = \arg \zeta^{(k)}$ is a multiple of $2\pi/n$ for every k. Let $\alpha = a_0 + a_1\zeta + \ldots + a_{l-1}\zeta^{l-1} \in \boldsymbol{Q}(\zeta)(a_k \in \boldsymbol{Q})$. Then, the kth conjugate of α in $\boldsymbol{Q}(\zeta)$ is given by $\alpha^{(k)} = a_0 + a_1\zeta^{(k)} + \ldots + a_{l-1}(\zeta^{(k)})^{l-1}$.

$1, \zeta, \ldots, \zeta^{l-1}$ are linearly independent over $\boldsymbol{Q}$ and so over $\boldsymbol{Z}$, the integral domain of real integers. These l complex numbers form an integral basis of $\boldsymbol{Z}(\zeta)$, the ring of algebraic integers in $\boldsymbol{Q}(\zeta)$. That is,

$$\boldsymbol{Z}(\zeta) = \{n_0 + n_1\zeta + \ldots + n_{l-1}\zeta^{l-1} | n_k \in \boldsymbol{Z}\}.$$

$\zeta^l, \zeta^{l-1}, \ldots, \zeta^{n-1}$ are given with integer coefficients in terms of the basis. Therefore, only l of n vertex vectors of a regular n-gon are linearly independent over $\boldsymbol{Q}$.

Let us introduce an l-dimensional complex row vector by $u = (1, \zeta, \ldots, \zeta^{l-1})$. Then, ζu is also a complex vector whose components belong to $\boldsymbol{Z}(\zeta)$. Accordingly, there exists one integer matrix $\mathbf{R}$ such that $\zeta u = u\mathbf{R}$. More precisely, we obtain $R_{i,i-1} = 1$ for $i = 1-(l-1)$ and $R_{i,l-1} = -c_i$ for $i = 0-(l-1)$ but other matrix elements vanish. Since ζ^{-1} belongs to $\boldsymbol{Z}(\zeta)$, $\mathbf{R}$ has to be a unimodular matrix. In fact, we can show easily that $\det \mathbf{R} = 1$. Obviously, the order of $\mathbf{R}$ is n; $\mathbf{R}^k \neq \mathbf{I}$ for $0 < k < n$ but $\mathbf{R}^n = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix in l dimensions. Note that $P_n(x)$ is the characteristic polynomial of $\mathbf{R}$: $\det(x\mathbf{I} - \mathbf{R}) = P_n(x)$. ζ and its conjugates $\zeta', \ldots, \zeta^{(l-1)}$ are different eigenvalues of $\mathbf{R}$, while u and its conjugates $u', \ldots, u^{(l-1)}$ with $u^{(k)} = (1, \zeta^{(k)}, \ldots, (\zeta^{(k)})^{l-1})$ are the corresponding left eigenvectors. Note that $u^{(k+m)}$ is the complex conjugate of $u^{(k)}$ for $k = 0, 1, \ldots, m-1$.

Let $\mathbf{U}$ be an $m \times l$ complex matrix whose kth row is given by $u^{(k)}$, $k = 0, 1, \ldots, m-1$. Then, $\mathbf{UR} = \mathbf{DU}$, where $\mathbf{D}$ is an $m \times m$ diagonal unitary matrix whose kth element is equal to $\zeta^{(k)}$. Let $\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{l-1}$ be the column vectors of $\mathbf{U}$. Then, they are m-dimensional complex vectors which are linearly independent over the real field $\mathbf{R}$; this follows from linear independence of $2m$ complex row vectors, $u^{(k)}$, $\bar{u}^{(k)}$, $k = 0, 1, \ldots, m-1$. If follows that

$$\{x_0\boldsymbol{\varepsilon}_0 + x_1\boldsymbol{\varepsilon}_1 + \ldots + x_{l-1}\boldsymbol{\varepsilon}_{l-1} | x_k \in \boldsymbol{R}\} \simeq \boldsymbol{C}^m \simeq \boldsymbol{C} \oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}$$

where the symbol $\simeq$ stands for an isomorphism relationship between vector spaces with Euclidean norms. Since $\boldsymbol{C}$ is isomorphous to E_2 (the Euclidean norm of $z \in \boldsymbol{C}$ is $|z|$), we can conclude that $\boldsymbol{C}^m \simeq E_l$. Therefore, we shall identify $\boldsymbol{C}$ with E_2 and $\boldsymbol{C}^m$ with E_l. We take $\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{l-1}$ as the basis vectors of $\boldsymbol{C}^m (\simeq E_l)$ though they are not orthonormal. Moreover, we denote by $L = L^{(n)}$ the l-dimensional real lattice generated by the basis vectors

$$L = \{n_0\boldsymbol{\varepsilon}_0 + n_1\boldsymbol{\varepsilon}_1 + \ldots + n_{l-1}\boldsymbol{\varepsilon}_{l-1} | n_k \in \boldsymbol{Z}\}. \tag{1}$$

Let us introduce a linear transformation ρ of $\boldsymbol{C}^m$ by $\rho(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{l-1}) = (\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{l-1})\mathbf{R}$. Then, ρ is an orthogonal transformation of $\boldsymbol{C}^m$; $\rho z = \mathbf{D}z$ for $z \in \boldsymbol{C}^m$, so that $|\rho z| = |z|$. Since $\mathbf{R}$ is a unimodular matrix, ρ is an element of the point symmetry of L. Note that each component in $\boldsymbol{C}^m = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}$ is an invariant subspace against ρ, which acts on the kth component as a multiplication of $\zeta^{(k)}$ (a rotation by θ_k).

Every element of $\bar{u}$ belongs to $\boldsymbol{Z}(\zeta)$, so that there exists an integer matrix $\mathbf{S}$ such that $\bar{u} = u\mathbf{S}$. It follows that $\mathbf{S}^2 = \mathbf{I}$ because $\bar{\bar{u}} = u$. Therefore, $\mathbf{S}$ is also a unimodular matrix. Obviously, $\bar{\mathbf{U}} = \mathbf{US}$. Accordingly, we can define an orthogonal transformation σ of $E_l \simeq \boldsymbol{C}^m$ by using $\mathbf{S}$; σ acts on each component in $\boldsymbol{C}^m$ as the complex conjugate operation and σ is a symmetry element of L.

The point group D_n is isomorphous to a finite group G generated by ρ and σ; G is a point symmetry group of L and it acts on the zeroth component of $\boldsymbol{C}^m$ as D_n. Note that each component in $\boldsymbol{C}^m$ is an irreducible invariant subspace against $G \simeq D_n$. Note also that the kth component of $z = n_0\varepsilon_0 + n_1\varepsilon_1 + \ldots + n_{l-1}\varepsilon_{l-1} \in L$ is equal to $n_0 + n_1\zeta^{(k)} + \ldots + n_{l-1}(\zeta^{(k)})^{l-1}$, which is nothing but the projection of z onto the kth subspace.

Let $\varepsilon_k = \rho^k\varepsilon_0$, $k = l, l+1, \ldots, n-1$. Then $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are lattice vectors with an equal length ($\sqrt{m}$) of L. They represent n nearest neighbours of the lattice site at the origin. Their projections onto the kth subspace in $\boldsymbol{C}^m$ are 1, $\zeta^{(k)}$, $(\zeta^{(k)})^2, \ldots, (\zeta^{(k)})^{n-1}$, which represent the vertices of the 'unit' regular n-gon in the relevant space; the n-complex numbers are rearranged to $1, \zeta, \ldots, \zeta^{n-1}$. Accordingly, we may call L an n-gonal lattice. The projection of L onto the first component in $\boldsymbol{C}^m$ is the 'standard (quasi-)lattice' in the terminology of Mermin *et al* (1987).

The arguments in the present section apply equally to the cases of crystallographic point symmetries, D_4 and D_6. In these cases, the multiplicities are equal to 1. $L^{(4)}$ and $L^{(6)}$ are the square and the triangular lattices, respectively.

If $n = 2p$ with p being a prime integer ($p \geqslant 5$), $L^{(n)}$ is an n-gonal lattice in $p-1$ dimensions. A decagonal lattice in 4D is discussed by Janssen (1986).

If $n = 2^k$, $k \geqslant 3$, then $l = 2^{k-1}$. We can show easily in this case that the basis vectors $\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{l-1}$ are orthogonal to each other. Hence, $L^{(n)}$ is a simple hypercubic lattice in l dimensions.

If $n = 2^k3^{k'}$ with k, $k' \geqslant 1$, then, $l = n/3$ and $m = n/6$. A two-dimensional lattice generated by $\boldsymbol{\varepsilon}_k$ and $\boldsymbol{\varepsilon}_{m+k}$ is a triangular lattice for $k = 0, 1, \ldots, m-1$ and different triangular lattices are orthogonal to each other in E_l, so that $L^{(n)}$ in this case is an l-dimensional hyperhexagonal lattice which is a direct proproduct of m identical triangular lattices (i.e. hexagonal lattices in 2D). More generally, if $n = 2^kp^{k'}$ with k, $k' \geqslant 1$ and $p(\geqslant 5)$ being a prime integer, $L^{(n)}$ is a direct product of $(m/2p)$ of identical $2p$-gonal lattices in $p-1$ dimensions.

3. A construction of an n-gonal quasilattice in 2D

We begin by decomposing $\boldsymbol{C}^m$ into the external and internal spaces as $\boldsymbol{C}^m = \boldsymbol{C} \oplus \boldsymbol{C}^{m-1}$, respectively, where $\boldsymbol{C}$ ($\simeq E_2$) is the zeroth invariant subspace against G ($\simeq D_n$) and $\boldsymbol{C}^{m-1}$ ($\simeq E_{l-2}$) is the orthogonal complement of $\boldsymbol{C}$ in $\boldsymbol{C}^m$ ($\simeq E_l$). $\boldsymbol{C}^{m-1}$ is also an invariant subspace against G but it is reducible unless $m = 2$ (or $l = 4$). G acts on $\boldsymbol{C}^{m-1}$ as a point group D'_n ($\simeq D_n$).

Let W be a finite domain in $\boldsymbol{C}^{m-1}$ and assume that it is invariant against D'_n. Then the following set of points in $\boldsymbol{C}$ is a two-dimensional quasilattice with point symmetry D_n (an n-gonal quasilattice):

$$L_Q(\phi, W) = \{Pz \mid z \in L \text{ and } P'z \in \boldsymbol{\phi} + W\} \tag{2a}$$

$$= \left\{ \sum_{k=0}^{l-1} n_k\zeta^k \,\middle|\, n'_k s \in \boldsymbol{Z} \text{ and } \sum_{k=0}^{l-1} n_k\boldsymbol{\alpha}_k \in \boldsymbol{\phi} + W \right\} \tag{2b}$$

where P and P' are projection operators to subspaces $\boldsymbol{C}$ and $\boldsymbol{C}^{m-1}$, respectively, $\boldsymbol{\phi}$ is an $(m-1)$-dimensional complex vector and $\boldsymbol{\alpha}_k = P'\varepsilon_k$, $k = 1, 2, \ldots, l-1$. Obviously, $L_Q(\boldsymbol{\phi}, W)$ is a subset of $\boldsymbol{Z}(\zeta)$. W is called the window and $\boldsymbol{\phi}$ the phase vector. Two quasilattices with a common W but different $\boldsymbol{\phi}$ belong to the same local-isomorphism

(LI) class (for the definition of LI class, see Levine and Steinhardt (1986)). We shall denote an LI class characterised by W as $\mathrm{C}(W)$. W has, by assumption, the centre of symmetry which coincides with the origin in $\boldsymbol{C}^{m-1}$.

Thus, an octagonal or 16-gonal quasilattice can be, for example, obtained from a simple hypercubic lattice in 4D (Ishihara *et al* 1988) or 8D, respectively. They are obtained, alternatively, as the dual lattices of a tetragrid and an octagrid, respectively (Niizeki 1988b). On the other hand, dodecagonal, 18-gonal and 24-gonal quasilattices are obtained from hyperhexagonal lattices in 4D, 6D and 8D, respectively (Niizeki and Mitani 1987). They are, alternatively, obtained as the dual lattices of double, triple and quadruple honeycomb grids (Stampfli 1986, Niizeki 1988a, Korepin *et al* 1988), respectively.

4. Self-similarity of an n-gonal quasilattice in 2D

Usual PV numbers are defined only for a real algebraic field (Salem 1983). In this paper, we shall extend the definition to the case of the cyclotomic field $\boldsymbol{Q}(\zeta)$, which is a complex field. A number τ in $\boldsymbol{Q}(\zeta)$ has $2m$ conjugates including itself, τ, $\tau', \ldots, \tau^{(2m-1)}$, but the second half of them are the complex conjugates of the first half, $\tau^{(k+m)} = \bar{\tau}^{(k)}$, $k = 0, 1, \ldots, m-1$. In particular, $\tau^{(k+m)} = \tau^{(k)}$, $k = 0, 1, \ldots, m-1$, if τ is real. Therefore, it is natural to define a PV number in $\boldsymbol{Q}(\zeta)$ as follows: $\tau \in \boldsymbol{Q}(\zeta)$ is a PV number if (i) $|\tau| > 1$ and (ii) $|\tau^{(k)}| < 1$ for $k = 1, 2, \ldots, m-1$.

If τ is a PV number and also a unit in $\boldsymbol{Q}(\zeta)$, then τ is a PV unit. In this case, let $\eta = \tau\tau' \ldots \tau^{(m-1)}$. Then η is a unit and $\eta\bar{\eta}(=\tau\tau' \ldots \tau^{(2m-1)})$ is a real integer. Therefore, we can conclude that $\eta\bar{\eta} = 1$ and $\eta = \zeta^k$ for some k.

Let τ be a PV unit in $\boldsymbol{Q}(\zeta)$. Then, there exists an integer matrix $\mathbf{T}$ satisfying $\tau u = u\mathbf{T}$ because $\tau \in \boldsymbol{Z}(\zeta)$. Moreover, $\mathbf{T}$ is a unimodular matrix because $\tau^{-1} \in \boldsymbol{Z}(\zeta)$. It follows that $\tau^{(k)} u^{(k)} = u^{(k)}\mathbf{T}$, $k = 1, 2, \ldots, m-1$, where $\tau^{(k)}$ is the kth conjugate of τ in $\boldsymbol{Q}(\zeta)$. Hence, $\mathbf{UT} = \mathbf{VU}$ with $\mathbf{V}$ being an m-dimensional diagonal matrix whose kth element is equal to $\tau^{(k)}$. The $\tau^{(k)}$ are eigenvalues of $\mathbf{T}$ and the $u^{(k)}$ are the corresponding left eigenvectors. Note that $\det \mathbf{T} = \eta\bar{\eta} = 1$. Since $\tau \in \boldsymbol{Z}(\zeta)$, τ is written as $\tau = j_0 + j_1\zeta + \ldots j_{l-1}\zeta^{l-1}$ with the j_k being integers. It follows that $\mathbf{T} = j_0\mathbf{I} + j_1\mathbf{R} + \ldots + j_{l-1}\mathbf{R}^{l-1}$.

Using the basis vectors $\boldsymbol{\varepsilon}_k$, we can define with $\mathbf{T}$ a linear transformation χ of $E_l \simeq \boldsymbol{C}^m$. χ is an automorphism of L because $\mathbf{T}$ is unimodular. Moreover, each component in $\boldsymbol{C}^m$ is an invariant subspace against χ, which acts on the kth supspace as a multiplication of $\tau^{(k)}$. It follows that χ acts on $\boldsymbol{C}$ in $\boldsymbol{C}^m = \boldsymbol{C} \oplus \boldsymbol{C}^{m-1}$ as a homogeneous similarity transformation (HST) characterised by τ, while it acts on $\boldsymbol{C}^{m-1}$ as a contractive linear transformation (CLT) because $|\tau^{(k)}| < 1$ for $k = 1, 2, \ldots, m-1$. We shall denote by χ' the restriction of χ onto $\boldsymbol{C}^{m-1}$. Note that χ is not an orthogonal transformation of $\boldsymbol{C}^m$ $(\simeq E_l)$ but a volume-conserving one. Note also that $\tau P = P\chi$ and $\chi' P' = P'\chi$.

Now, the conditions $z \in L$ and $P'z \in \boldsymbol{\phi} + W$ are equivalent to $\chi z \in L$ and $\chi' P' z \in \chi'\boldsymbol{\phi} + \chi' W$, respectively. Then, it follows from $(2a)$ that

$$L_Q(\boldsymbol{\phi}, W) = \{Pz \mid \chi z \in L \text{ and } \chi' P' z \in \chi'\boldsymbol{\phi} + \chi'\boldsymbol{\phi} + \chi' W\}$$

so that

$$\tau L_Q(\boldsymbol{\phi}, W) = \{P\chi z \mid \chi z \in L \text{ and } P'\chi z \in \chi'\boldsymbol{\phi} + \chi' W\}.$$

The right-hand side of the second equality is equal tot $L_Q(\chi'\boldsymbol{\phi}, \chi'W)$ because χ is an automorphism of L. Replacing $\boldsymbol{\phi}' = (\chi')^{-1}\boldsymbol{\phi}$ in place of $\boldsymbol{\phi}$ we obtain $\tau L_Q(\boldsymbol{\phi}', W) = L_Q(\boldsymbol{\phi}, \chi'W)$. This means that two different LI classes $\mathrm{C}(W)$ and $\mathrm{C}(\chi'W)$ are in fact similar.

Since χ' is a CLT, $(\chi')^N$ tends to a null mapping as $N \to \infty$. Therefore, $(\chi')^N W \subset W$ for some N. Actually, we may take $N=1$ if W is an $(l-2)$-dimensional sphere in $\boldsymbol{C}^{m-1}$ $(\simeq E_{l-2})$ or if it is sufficiently close to such a sphere. If $N \neq 1$, we may redefine τ^N to be τ because τ^N is also a PV unit. Therefore, we may assume that $\chi'W \subset W$. Then, $L_Q(\boldsymbol{\phi}, \chi'W)$ is a sublattice of $L_Q(\boldsymbol{\phi}, W)$. Thus, a sublattice of an n-gonal quasilattice $L_Q(\boldsymbol{\phi}, W)$ coincides with a transformation by τ of another quasilattice $L_Q(\boldsymbol{\phi}', W)$ which belongs to the same LI class as that of $L_Q(\boldsymbol{\phi}, W)$. This means that $L_Q(\boldsymbol{\phi}, W)$ has a self-similarity characterised by τ; the relevant transformation is a combined operation of a dilatation by $|\tau|$ with a rotation by arg τ (if arg $\tau \neq 0$). The inflation rule of the self-similarity of $L_Q(\boldsymbol{\phi}, W)$ is to contract the window W to $\chi'W$.

We shall call τ the complex self-similarity ratio, though it is sometimes real. An n-gonal quasilattice always has a self-similarity for any n because every algebraic field includes an infinite number of PV units (Salem 1983).

If τ_1 and τ_2 are PV units, then so is $\tau_1\tau_2$. Therefore, the set of all the PV units in $\boldsymbol{Q}(\zeta)$ form a commutable semigroup, which we shall call the self-similarity semigroup (SSSG).

5. The self-similarity ratios of *n*-gonal quasilattices whose multiplicities are less than 5

All the units in $\boldsymbol{Q}(\zeta)$ form an Abelian group $\mathscr{U}$ with respect to the multiplication, which is called the group of units. The cyclic group $\mathrm{C}_n = \{1, \zeta, \ldots, \zeta^{n-1}\}$ is a subgroup of $\mathscr{U}$; each unit in C_n is related to a rotational symmetry of an n-gonal quasilattice. The absolute values of other units than those in C_n are not equal to one. According to the unit theorem due to Dirichlet, $\mathscr{U}$ is a direct product of C_n and a free Abelian group generated by $(m-1)$ fundamental units.

It is obvious that the SSSG is embedded in $\mathscr{U}$. The fundamental units of $\boldsymbol{Q}(\zeta)$ may not all be PV units. In several cases presented later on, we can take PV units as the fundamental units. Then they generate $\mathscr{U}$ but they do not usually generate the SSSG because taking the inverse of an element is not an allowed operation in a semigroup. If $m=2$, there exists only one fundamental unit, which is assumed to be a PV unit. Then, a relevant quasilattice has only one scale of self-similarity. In contrast, a quasilattice in the case of $m>2$ may have two or more independent scales of self-similarity.

A maximal real subfield of $\boldsymbol{Q}(\zeta)$ with $\zeta = \zeta_n$, is $\boldsymbol{Q}(\xi)$, where $\xi = \zeta + \zeta^{-1} = 2\cos(2\pi/n)$ is an algebraic integer of order m. A real unit in $\boldsymbol{Q}(\zeta)$ is also a unit in $\boldsymbol{Q}(\xi)$. The group of units of $\boldsymbol{Q}(\xi)$ has also $(m-1)$ fundamental units, which are, however, not always the fundamental units of $\boldsymbol{Q}(\zeta)$.

If τ is a PV unit, so is $\zeta^k\tau$ for any k. Moreover, $\bar{\tau}$ is also a PV unit. Therefore, we assume hereafter that $0 \leqslant \arg \tau \leqslant \pi/n$. In fact, it is known that arg τ can take 0 or π/n but no other values. It follows that a complex PV unit takes the form $\tau = \sqrt{\tau_{\mathrm{r}}}\exp(\mathrm{i}\pi/n)$ with τ_{r} $(=\zeta^{-1}\tau^2 = |\tau|^2)$ being a real PV unit.

A fundamental unit in $\boldsymbol{Q}(\zeta)$ is written as $(1-\zeta^{(k)})/(1-\zeta^{(k')})$, where $0 \leqslant k, k' < m$. Some of the units of this form can be PV units.

5.1. The case of $m=2$

In the case of $m=2$, the internal space is two dimensional as is the external one. This is realised only when $n=8$, 10 and 12. Then $\boldsymbol{Q}(\zeta_n)$ (or $\boldsymbol{Q}(\xi_n)$) has only one fundamental unit. $\boldsymbol{Q}(\xi_n)$ is a real quadratic field whose fundamental unit is $1+\sqrt{2}$, $(1+\sqrt{5})/2$ or $2+\sqrt{3}$ according as $n=8$, 10 or 12, respectively. The first two units are also fundamental units of $\boldsymbol{Q}(\zeta_8)$ and $\boldsymbol{Q}(\zeta_{10})$, respectively. However, $2+\sqrt{3}$ is not a fundamental unit of $\boldsymbol{Q}(\zeta)$ with $\zeta=\zeta_{12}$ but is factorised as $\zeta^{-1}\tau^2$ with $\tau=1+\zeta$ being a fundamental unit. Note that $\arg\tau$ $(=\pi/12)$ is not a multiple of $\pi/6$ $(=2\pi/12)$, so that $\tau_{\mathrm{P}}=|\tau|$ $(=2\cos(\pi/12)=(\sqrt{3}+1)/\sqrt{2}\doteqdot 1.973)$, 'the platinum ratio', does not belong to $\boldsymbol{Q}(\xi_{12})=\boldsymbol{Q}(\sqrt{3})$. The conjugate of τ is $\tau'=1-\bar{\zeta}$.

$1+\sqrt{2}$, $(1+\sqrt{5})/2$ and $1+\zeta_{12}$ are PV units and, accordingly, they are the complex self-similarity ratios of octagonal, decagonal and dodecagonal quasilattices, respectively. The inflation of an octagonal quasilattice by $1+\sqrt{2}$ is given by Grünbaum and Shephard (1986) and those of dodecagonal ones by $1+\zeta_{12}$ by Stampfli (1986) and Niizeki and Mitani (1987). We show in figure 1 an inflation of a decagonal quasilattice by $(1+\sqrt{5})/2$; the quasilattice is obtained from a decagonal lattice in 4D by using a window of a regular decagon whose vertices are at $1, \zeta, \zeta^2, \ldots, \zeta^9$ in the internal space.

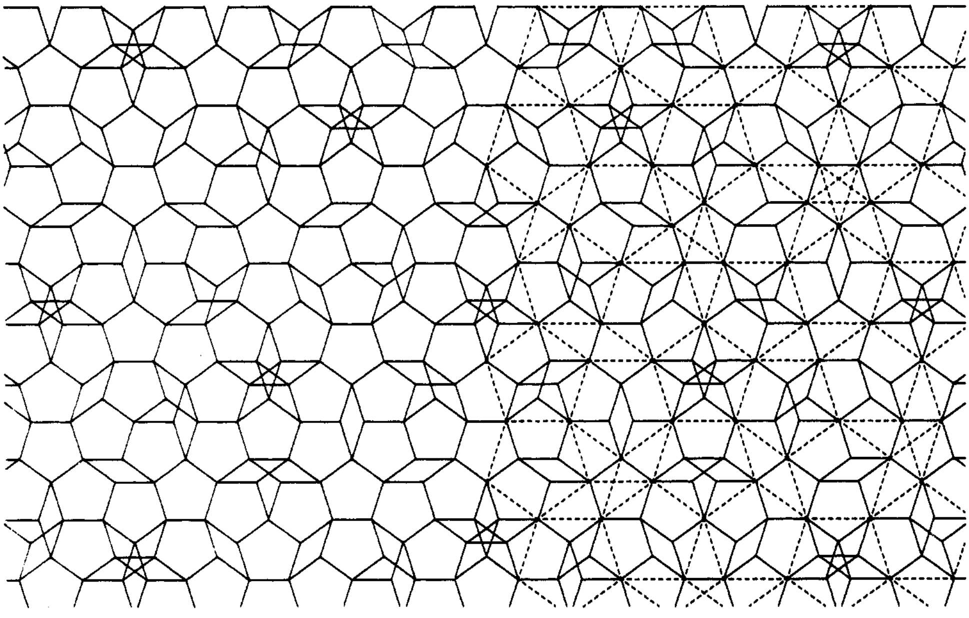
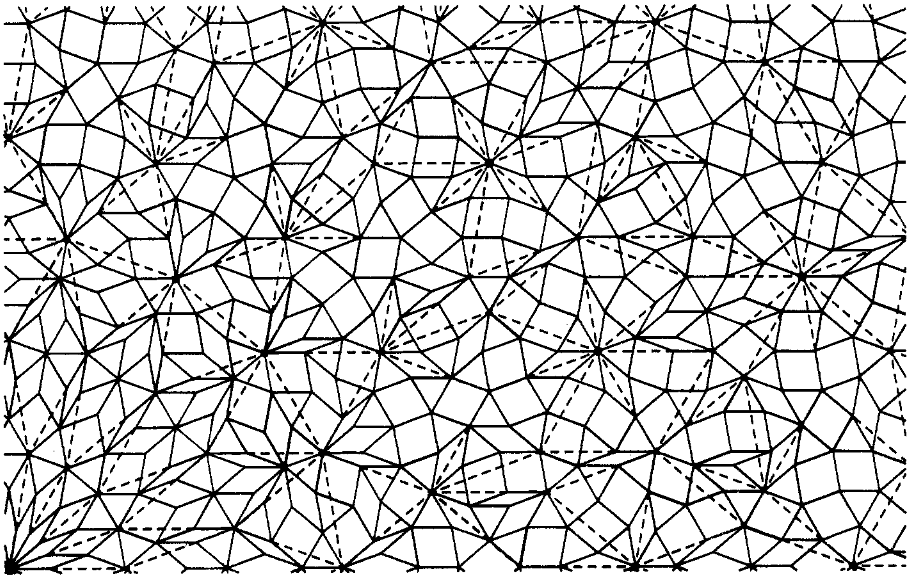


Figure 1. A decagonal quasilattice obtained from a decagonal lattice in 4D (full lines) and its inflated version (broken lines) with the scale $\tau_{\mathrm{G}}=(1+\sqrt{5})/2$. The bonds are drawn between 'arithmetic neighbour' pairs of sites.

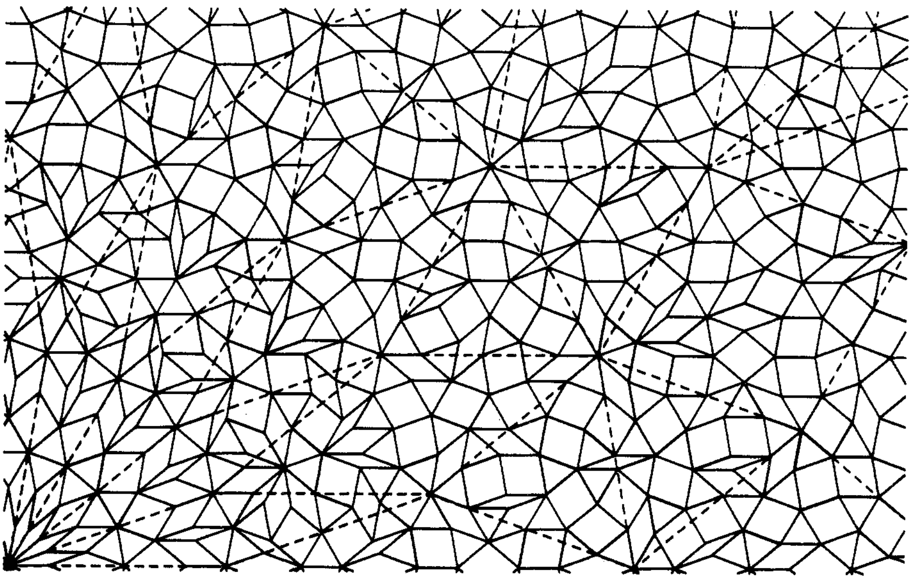
5.2. The case of $m=3$

In the case of $m=3$, the internal space is four dimensional. This occurs only when $n=14$ and 18. $\boldsymbol{Q}(\zeta)$ with $\zeta=\zeta_{14}$ or ζ_{18} has two fundamental units.

In the case of $n=14$, $\tau_1=1+\zeta^2+\zeta^{-2}$ $(=1+2\cos(2\pi/7)\doteqdot 2.247)$ and $\tau_2=\tau_1^2-1$ $(\doteqdot 4.049)$ are fundamental PV units; τ_2 is identical to the one reported by Gähler (1986)



(a)



(b)

Figure 2. An 18-gonal quasilattice (quasiperiodic tiling) obtained as a dual to a triple honeycomb grid and its inflated versions with the scales (a) $\tau_1 = 1 + 2\cos(\pi/9)$ and (b) $\tau_2 = \tau_1(\tau_1 - 1)$. The two scales are multiplicatively independent though algebraically dependent of each other. The tiling consists of four kinds of tiles, i.e an equilateral triangle and three rhombi whose acute inner angles are 20°, 40° and 80°.

as the self-similarity ratio of a 14-gonal quasilattice. τ_1 and τ_2 are independent scales because $\log \tau_1/\log \tau_2$ is irrational. The third smallest PV unit, $\tau_3 = \tau_1(\tau_1 + 1)$ ($\doteqdot 7.296$), is related to τ_1 and τ_2 as $\tau_3 = \tau_2^2(\tau_1)^{-1}$, which is, however, not a relationship as a semigroup.

In the case of $n = 18$, $\tau_1 = 1 + \zeta + \zeta^{-1}$ ($= 1 + 2\cos(\pi/9) \doteqdot 2.879$) and $\tau_2 = \tau_1^2 - \tau_1$ ($\doteqdot 5.411$) are fundamental PV units. We show in figures 2(*a*) and 2(*b*) an 18-gonal quasilattice and its inflated versions by the two independent scales τ_1 and τ_2, respectively.

5.3. *The case of* $m = 4$

The multiplicity takes $m = 4$ for $n = 16$, 20, 24, and 30 and the relevant algebraic field in each case has three fundamental units. In the present case we present only the PV unit with the smallest absolute value for each case. They are real for $n = 16$ and 30; $\tau = \xi^3 + 2\xi^2 - 1$ with $\xi = \xi_{16}$ for $n = 16$ and $\tau = \xi^3 + \xi^2 - 2\xi - 1$ with $\xi = \xi_{30}$ for $n = 30$. They are biquadratic algebraic integers; $\tau_{16} = \tau_s(\tau_s + \sqrt{2 + 2\tau_S})$ ($\doteqdot 12.14$) and $\tau_{30} = \tau_G(\tau_G^2 + \sqrt{6 + 3\tau_G})/2$ ($\doteqdot 4.783$), where $\tau_S = 1 + \sqrt{2}$ and $\tau_G = (1 + \sqrt{5})/2$ are silver and golden ratios, respectively.

On the other hand, the PV units with the smallest magnitudes are complex for $n = 20$ and 24; for $n = 20$, $\tau = \zeta^{-2}(1 + \zeta + \ldots + \zeta^5)$ ($= \zeta^4(1 + \zeta^4)/(\zeta - 1)$), so that $\arg \tau = \pi/20$ and $|\tau| = \cos(\pi/5)/\sin(\pi/20) \doteqdot 5.172$ and for $n = 24$, $\tau = (1 + \zeta)(\zeta + \zeta^{-1})(= \zeta^7/(\zeta - 1))$, so that $\arg \tau = \pi/24$ and $|\tau| = |1 - \zeta|^{-1} = [2\sin(\pi/24)]^{-1} = 3.831$. We show in figure 3 a 24-gonal quasilattice obtained from a hyperhexagonal lattice in 8D and its inflated version by τ. Contrary to the case of τ being real, the 24 directions of the 'bonds' in the inflated lattice are different from those in the original lattice; the two systems of directions are rotated by $\pi/24$ relative to each other.

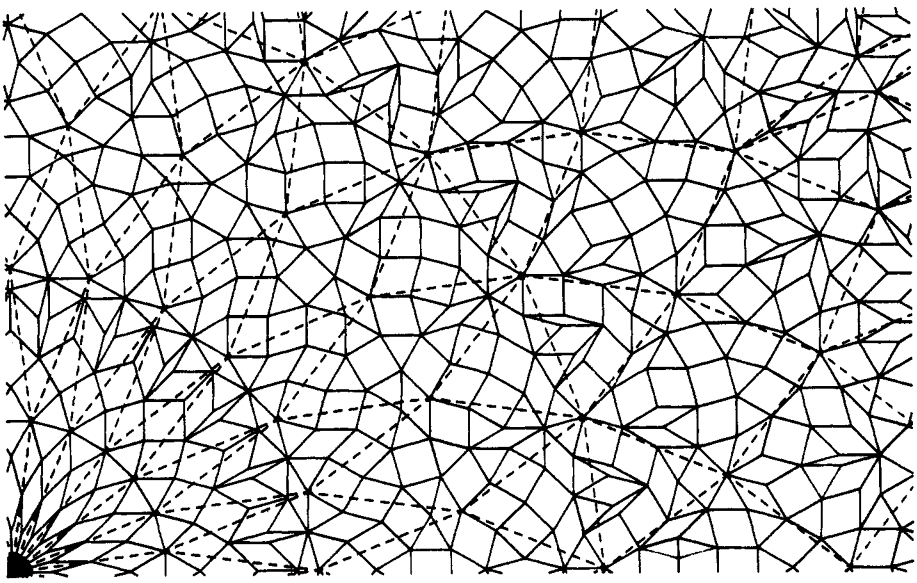


Figure 3. A 24-gonal quasilattice obtained as a dual to a quadruple honeycomb grid and its inflated version by the complex self-similarity ratio $\tau = (1 + \zeta)(\zeta + \zeta^{-1})$. The directions of bonds in the inflated lattice are rotated by $\pi/24$ from those of the original one, because τ is essentially complex.

6. The self-similarity in the diffraction pattern of an n-gonal quasilattice

The reciprocal space (the wavenumber space) of $E_l \simeq C^m = C \oplus C \oplus \ldots \oplus C$ is also an l-dimensional Euclidean space $E_{l,*} \simeq (C_*)^m = C_* \oplus C_* \oplus \ldots \oplus C_*$ with $C_* \simeq E_{2,*}$ being a complex plane. It is decomposed into the reciprocal internal space and the external counterpart as $C_*^m = C_* \oplus C_*^{m-1}$. The Fourier transform (FT) of $L_Q(\boldsymbol{\phi}, W)$ is given as the convolution of the FT of L and that of the window function (Zia and Dallas 1985, Katz and Duneau 1986). The FT of L is a sum of delta functions whose supports are the lattice vectors of L_*, the reciprocal lattice of L. Consequently, a reciprocal lattice vector of $L_Q(\boldsymbol{\phi}, W)$ is represented by a complex number in C_* of the form $\beta = Pg$ with $g \in L_*$, while the intensity is the absolute value of the Fourier component, $W_*(\xi)$, of the window function with respect to the wavevector $\xi = P'g$ in C_*^{m-1}.

The automorphism χ of L yields an automorphism χ_* of L_*. χ_* acts on the reciprocal basis vectors of L_* as $\chi_*(\varepsilon_0^*, \varepsilon_1^*, \ldots, \varepsilon_{m-1}^*) = (\varepsilon_0^*, \varepsilon_1^*, \ldots, \varepsilon_{m-1}^*)({}^t\mathbf{T})^{-1}$, where ${}^t\mathbf{T}$ denotes the transpose of $\mathbf{T}$. Therefore, $(\chi_*)^{-1}$ acts on C_* as a HST characterised by $\bar{\tau}$ and on C_*^{m-1} as a CLT.

If $g \in L_*$ is transformed by $(\chi_*)^{-1}$, $\beta = Pg$ is inflated as $\bar{\tau}\beta$, while $\xi = P'g$ is contracted. Thus, a series of Bragg spots with increasing intensities appear in the reciprocal space at $(\bar{\tau})^k \beta_0$, $k = 0, 1, 2, \ldots$, where β_0 represents the starting spot. This has been observed by Elser (1985) and Levine and Steinhardt (1986) in their studies of an icosahedral quasilattice in 3D. We can observe this in the diffraction pattern of the decagonal lattice in figure 1 as presented in figure 4. An example of the case where τ is a complex number can be found in Niizeki and Mitani (1987).

7. Discussions

As noted in § 4, the inflation rule of an n-gonal quasilattice is to narrow the window W to $\chi' W$. If W is contrariwise extended to $(\chi')^{-1} W$, new lattice points are introduced and the resulting quasilattice is similar to the original one with the ratio τ^{-1} because $L_Q(\boldsymbol{\phi}, \chi' W) = \tau^{-1} L_Q(\chi' \boldsymbol{\phi}, W)$. Thus, an n-gonal quasilattice has self-similarity also with respect to a deflation. Note, however, that the SSSG can be raised to a self-similarity group only when $m = 2$.

The n-gonal lattice L is a Bravais lattice but it includes many non-Bravais lattices as its sublattices. Some of them are invariant against G and they yield new LI classes of n-gonal quasilattices. The Penrose lattice can be constructed in this way from a non-Bravais sublattice of the decagonal lattice in 2D (Janssen 1986). Also, several dodecagonal quasilattices are obtained from non-Bravais sublattices of the hyper-hexagonal lattice in 4D (Niizeki 1988a). It is, however, complicated to show self-similarity of these 'non-Bravais-type quasilattices' (de Bruijn 1981, Niizeki 1988a) and a full argument on this subject will be presented in the following paper (Niizeki 1989).

Watanabe *et al* (1987) showed that an octagonal quasilattice has an inflation with the ratio $2+\sqrt{2}$. This number is a PV number in $Q(\sqrt{2})$ ($\xi_8 = \sqrt{2}$) but not a unit. It can be shown also that dodecagonal quasilattice has a self-similarity characterised by $1+\sqrt{3}$, which is a non-unit PV number in $Q(\sqrt{3})$ ($\xi_{12} = \sqrt{3}$). The feature of self-similarity characterised by a non-unit PV number will be discussed in a separate paper.

All the lattice points in $L_Q(\boldsymbol{\phi}, W)$ are located on parallel lattice lines which are normal to the direction being parallel to 1 (Katz and Duneau 1986); the lattice lines

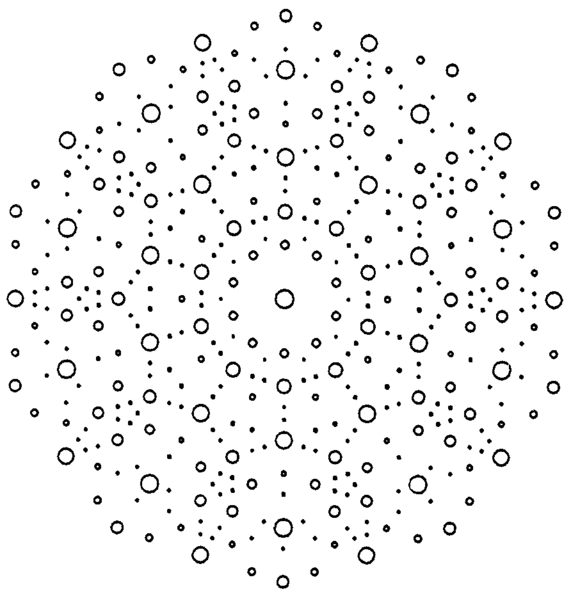


Figure 4. The diffraction pattern of a decagonal quasilattice in figure 1. The intensity of a spot is proportional to the area of the relevant circle. Series of spots with increasing intensities are arranged radially; the ratio of the wavenumbers between an adjacent pair in a series is equal to the golden ratio.

are arranged quasiperiodically along this direction. The same is true along other $(n/2-1)$ directions, ζ^k, $k=1,2,\ldots,(n/2-1)$. A superposition of these lattice lines yields a Pleasants n-gonal quasiperiodic pattern in 2D. The details of this subject will be presented elsewhere.

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